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Additive Property of Cube Root Expectation

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Abstract - In a recent study, one concept of expectation termed as cube root expectation had been introduced and its definition was formulated in the case of random variables (both discrete and continuous). One property of this expectation, termed here as additive property of cube root expectation, has here been derived in the case of random variable and in the case of function of random variable. This article describes the derivation of this property.

Keywords: Random Variable, Function, Cube Root Expectation, Additive Property.

1. INTRODUCTION

Expectation of a random variable, which had been introduced as the theoretical average of the possible values assumed by the variable. more specifically as the weighted average of its all possible values with their respective probabilities as the corresponding weights, was originally defined on the basis of weighted arithmetic mean terming it as mathematical expectation [4 , 16 , 17 , 20 , 24 - 26] which later was termed as arithmetic expectation [7 , 9]. Expectation, which plays a significant role in the theory of statistics, is the basic concept of unbiasedness of estimators in the theory of statistical estimation [14 , 15 , 21 , 23].

Later on, attempt was made on introducing and defining the concept of expectation systematically and consequently four concepts of expectation were introduced and defined on the basis of arithmetic mean [2, 5 , 6], geometric mean [2, 5 , 6], harmonic mean [2, 5 , 6], quadratic mean [3 , 6 ,13 , 18 , 19] and cubic mean [6 , 12 , 22] which were respectively termed as arithmetic expectation [7 , 9], geometric expectation [7 , 9], harmonic expectation [7 , 9], quadratic expectation [8 , 9] and cubic expectation [10]. It was found that arithmetic expectation was nothing new but the mathematical expectation defined earlier [3]. Recently, one more concept of expectation termed as cube root expectation of random variable was introduced with formulating its definition and deriving some of its elementary properties [11]. However, more properties are yet to be studied. Attempt has here been made on searching for if cube root expectation possesses some property of additive nature. One property of this expectation, termed here as additive property of cube root expectation, has here been derived in the case of random variable and also in the case of function of random variable. This article describes the derivation of this property.

2. CUBE ROOT EXPECTATION

Let X be a real valued random variable assuming values whose cube roots exist.
Then is cube root expectation [11] defined as follows:
If X is a finite discrete random variable assuming values

$$x_1 , x_2 , \ldots\ldots , x_N$$

with probabilities

$$p_1 , p_2 , \ldots\ldots , p_N ,$$

respectively, then the cube root expectation of X, denoted by ECR(X), is defined by

$$E_{CR}(X) = (\sum_{i=1}^{N} p_i x_i^{\frac{1}{3}})^3 \quad (2.1)$$

If X is a denumerable (countable infinite) discrete random variable assuming values

$$x_1 , x_2 , \ldots\ldots\ldots\ldots$$

with probabilities

$$p_1 , p_2 , \ldots\ldots\ldots\ldots$$

respectively, then ECR(X) is defined by

$$E_{CR}(X) = (\sum_{i=1}^{\infty} p_i x_i^{\frac{1}{3}})^3 \quad (2.2)$$

provided $\sum_{i=1}^{\infty} p_i x_i^{\frac{1}{3}}$ exists

Again, if X is a continuous random variable assuming values in any of the intervals

$$(a , b) \text{ or } [a , b) \text{ or } (a , b] \text{ or } [a , b]$$

with finite or infinite a & b, following probability density function f(x),
then $E_{CR}(X)$ is defined by

$$E_{CR}(X) = \{\int_a^b x^{\frac{1}{3}} f(x)dx\}^3 \quad (2.3)$$

Similarly, let $\varphi(X)$ be a function X.

If X is finite discrete then the cube root expectation of $\varphi(X)$, denoted by ECR{$\varphi(X)$}, is defined by

$$E_{CR}\{\varphi(X)\} = [\sum_{i=1}^{N} p_i \{\varphi(x_i)\}^{\frac{1}{3}}]^3 \quad (2.4)$$

If X is denumerable discrete, ECR{$\varphi(X)$} is defined by

$$E_{CR}\{\varphi(X)\} = [\sum_{i=1}^{\infty} p_i \{\varphi(x_i)\}^{\frac{1}{3}}]^3 \quad (2.5)$$

provided $\sum_{i=1}^{\infty} p_i \{\varphi(x_i)\}^{\frac{1}{3}}$ exists.

Again, if X is a continuous, $E_{CR}\{\varphi(X)\}$ is defined by

$$E_{CR}\{\varphi(X)\} = [\int_a^b \{\varphi(x)\}^{\frac{1}{3}} f(x)dx]^3 \quad (2.6)$$

Note (2.1):
Arithmetic expectation [7 , 9] is defined as follows:
Arithmetic expectation of X, denoted by $E_A(X)$, is defined by

$$E_A(X) = \sum_{i=1}^{N} p_i x_i \quad (2.7)$$

when X is finite discrete
and by

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$$E_A(X) = \sum_{i=1}^{\infty} p_i x_i \quad (2.8)$$

when X is denumerable discrete

provided $\sum_{i=1}^{\infty} p_i x_i$ exists

and by

$$E_A(X) = \int_a^b x.f(x)dx \quad (2.9)$$

when X is continuous [3 , 4 , 6].

Similarly, the arithmetic expectation of φ(X), denoted by EA{φ(X)}, is defined by

$$E_A\{\varphi(X)\} = \sum_{i=1}^{N} p_i.\varphi(x_i) \quad (2.10)$$

when X is finite discrete

and by

$$E_A\{\varphi(X)\} = \sum_{i=1}^{\infty} p_i.\varphi(x_i) \quad (2.11)$$

when X is denumerable discrete

provided $\sum_{i=1}^{\infty} p_i.\varphi(x_i)$ exists

and by

$$E_A\{\varphi(X)\} = \int_a^b \varphi(x)f(x)dx \quad (2.12)$$

when X is continuous.

Note (2.2):

Definitions of arithmetic expectation and cube root expectation implies that

$$E_{CR}(X) = \{E_A(X^{\frac{1}{3}})\}3 \quad (2.13)$$

$$\& \quad E_{CR}\{\varphi(X)\} = E_A[\{\varphi(X)\}^{\frac{1}{3}}]3 \quad (2.14)$$

i.e. the cube root expectation of a random variable X is the cube of the arithmetic expectation of its cube root and also the cube root expectation of a function φ(X) of a random variable X is the cube of the arithmetic expectation of the cube root of φ(X).

3. ADDITIVE PROPERTY IN THE CASE OF RANDOM VARIABLE

Let

$$X_1, X_2, \ldots\ldots, X_k$$

be k random variables whose arithmetic expectations and cube root expectations exist.

Then from the definitions of EA(X) and ECR(X),

$$E_{CR}(X_1) = \{E_A(X_1^{\frac{1}{3}})\}3, E_{CR}(X_2) = \{E_A(X_2^{\frac{1}{3}})\}3, \ldots\ldots, E_{CR}(X_k) = \{E_A(X_k^{\frac{1}{3}})\}3$$

or equivalently

$$\{E_{CR}(X_1)\}^{\frac{1}{3}} = E_A(X_1^{\frac{1}{3}}), \{E_{CR}(X_2)\}^{\frac{1}{3}} = E_A(X_2^{\frac{1}{3}}), \ldots\ldots, \{E_{CR}(X_k)\}^{\frac{1}{3}} = E_A(X_k^{\frac{1}{3}})$$

Replacing

$$X_1 , X_2 , \ldots\ldots , X_k$$

respectively by

$$X_1^3 , X_2^3 , \ldots\ldots , X_k^3$$

it is obtained that

$$E_{CR}(X_1^3) = \{E_A (X_1)\}3 , E_{CR}(X_2^3) = \{E_A (X_2)\}3 , \ldots\ldots , E_{CR}(X_k^3) = \{E_A (X_k)\}3$$

or equivalently

$$E_A(X_1) = \{E_{CR}(X_1^3)\}^{\frac{1}{3}} , E_A(X_2) = \{E_{CR}(X_2^3)\}^{\frac{1}{3}} , \ldots\ldots , E_A(X_k) = \{E_{CR}(X_k^3)\}^{\frac{1}{3}}$$

Replacing X_1 by $X_1 + X_2 + \ldots\ldots + X_k$ in the equation

$$E_{CR}(X_1^3) = \{E_A (X_1)\}3$$

it is obtained that

$$E_{CR}\{(X_1 + X_2 + \ldots\ldots + X_k)3\} = \{E_A (X_1 + X_2 + \ldots\ldots + X_k)\}3$$

Additive property of arithmetic expectation [15 , 18 , 21] implies that

$$E_A(X_1 + X_2 + \ldots\ldots + X_k) = E_A(X_1) + E_A(X_2) + \ldots\ldots + E_A(X_k)$$

Therefore,

$$E_{CR}\{(X_1 + X_2 + \ldots\ldots + X_k)3\} = \{E_A(X_1) + E_A(X_2) + \ldots\ldots + E_A(X_k)\}3$$

which implies,

$$[E_{CR}\{(X_1 + X_2 + \ldots\ldots\ldots + X_k)3\}]^{\frac{1}{3}} = \{E_{CR}(X_1 3)\}^{\frac{1}{3}} + \{E_{CR}(X_2 3)\}^{\frac{1}{3}} + \ldots\ldots\ldots$$

$$\ldots + \{E_{CR}(X_k 3)\}^{\frac{1}{3}} \qquad (3.1)$$

This can be regarded as the additive property of cube root expectation.

In particular, if X & Y are two random variables then

$$[E_{CR}\{(X + Y)3\}]^{\frac{1}{3}} = \{E_{CR}(X^3)\}^{\frac{1}{3}} + \{E_{CR}(Y^3)\}^{\frac{1}{3}} \qquad (3.2)$$

4. ADDITIVE PROPERTY IN THE CASE OF FUNCTION OF RANDOM VARIABLE

Let

$$\phi 1(X_1) = \phi 1 , \phi 2(X_1) = \phi 2 , \ldots\ldots\ldots , \phi k(X_k) = \phi k$$

be k real valued functions whose arithmetic expectations and cube root expectations exist.

Then from the definitions of $E_A\{\varphi(X)\}$ and $E_{CR}\{\varphi(X)\}$,

$$E_{CR}(\phi 1) = \{E_A (\phi_1^{\frac{1}{3}})\}3 , E_{CR}(\phi_2) = \{E_A (\phi_2^{\frac{1}{3}})\}3 , \ldots\ldots , E_{CR}(\phi_k) = \{E_A (\phi_k^{\frac{1}{3}})\}3$$

or equivalently

$$\{E_{CR}(\phi 1)\}^{\frac{1}{3}} = E_A(\phi_1^{\frac{1}{3}}) , \{E_{CR}(\phi_2)\}^{\frac{1}{3}} = E_A(\phi_2^{\frac{1}{3}}) , \ldots\ldots , \{E_{CR}(\phi_k)\}^{\frac{1}{3}} = E_A(\phi_k^{\frac{1}{3}})$$

Replacing

$$\phi 1 , \phi_2 , \ldots\ldots , \phi_k$$

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respectively by

$$\phi_1^{\,3}, \phi_2^{\,3}, \ldots\ldots, \phi_k^{\,3}$$

it is obtained that

$$E_{CR}(\phi_1^{\,3}) = \{E_A(\phi 1)\}3, E_{CR}(\phi_2^{\,3}) = \{E_A(\phi_2)\}3, \ldots\ldots, E_{CR}(\phi_k^{\,3}) = \{E_A(\phi_k)\}3$$

or equivalently

$$E_A(\phi 1) = \{E_{CR}(\phi_1^{\,3})\}^{\frac{1}{3}}, E_A(\phi_2) = \{E_{CR}(\phi_2^{\,3})\}^{\frac{1}{3}}, \ldots\ldots, E_A(\phi_k) = \{E_{CR}(\phi_k^{\,3})\}^{\frac{1}{3}}$$

Now, $E_{CR}(\phi_1^{\,3}) = \{E_A(\phi 1)\}3$ implies,

$$E_{CR}\{(\phi 1 + \phi_2 + \ldots\ldots + \phi_k)3\} = \{E_A(\phi 1 + \phi_2 + \ldots\ldots + \phi_k)\}3$$

Additive property of arithmetic expectation [7 , 9] implies that

$$E_A(\phi 1 + \phi_2 + \ldots\ldots + \phi_k) = E_A(\phi 1) + E_A(\phi_2) + \ldots\ldots + E_A(\phi_k)$$

Therefore,

$$E_{CR}\{(\phi 1 + \phi_2 + \ldots\ldots + \phi_k)3\} = \{E_A(\phi 1) + E_A(\phi_2) + \ldots\ldots + E_A(\phi_k)\}3$$

which implies,

$$[E_{CR}\{(\phi 1 + \phi_2 + \ldots\ldots\ldots + \phi_k)3\}]^{\frac{1}{3}} = \{E_{CR}(\phi_1^{\,3})\}^{\frac{1}{3}} + \{E_{CR}(\phi_2^{\,3})\}^{\frac{1}{3}} + \ldots\ldots\ldots$$
$$\ldots + \{E_{CR}(\phi_k^{\,3})\}^{\frac{1}{3}} \qquad (4.1)$$

This can be regarded as additive property of cubic expectation in the case of finite number of functions of random variables.

In particular, if

$$\phi(X) = \phi_X \;\&\; \psi(Y)\} = \psi_Y$$

are two real valued functions of random variables X & Y respectively such that their arithmetic expectations and cube root expectations exist then

$$[E_{CR}\{(\phi_X + \psi_Y)3\}]^{\frac{1}{3}} = \{E_{CR}(\phi_X^{\,3})\}^{\frac{1}{3}} + \{E_{CR}(\psi_Y^{\,3})\}^{\frac{1}{3}} \qquad (4.2)$$

5. CONCLUSION

The property of cube root expectation, termed here as additive property, obtained above can be summarized in the form of a theorem mentioned as follows:

Theorem:

Cube root of cube root expectation of cube of the sum of a finite number of random variables is the sum of the cube roots of individual cube root expectations of the cubs of the variables provided all these cube root expectations exist.

and

Cube root of cube root expectation of cube of the sum of a finite number of functions of random variable is the sum of cube roots of individual cube root expectations of the cubes of the functions provided all these cube root expectations exist.

This property/theorem describes a relationship between

(1) the cube root expectation of sum of a finite number of random variables and the individual cube root expectations of the variables

and

(2) the cube root expectation of sum of a finite number of functions of random variable and the individual cube root expectations of the functions.

It is to be noted that there may exist more properties of cube root expectation which are still unknown. This leads to a problem of further research. At this stage, one task of researcher is to study of searching for the unknown properties of cube root expectation. One more task of researcher is to make attempt to searching for relationship between cubic root expectation and each of arithmetic expectation, geometric expectation, harmonic expectation, quadratic expectation and cubic expectation.

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